The Phenomena of Chaotic Dynamics

Chaos is an eternal human truth that we all know: small disturbances or minor events lead to large events. In Dostoevsky's *The Brothers Karamazov* (1879) an old priest relates an important truth:

"A touch in one place sets up movement at the other end of the Earth."

In 1972 Lorenz asked:

"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

Benjamin Franklin described how a cascade of increasingly important events might emerge from a chance event:

"For the want of a nail the shoe was lost
For the want of a shoe the horse was lost.
For the want of a horse a rider was lost.
For the want of a rider the battle was lost.
For the want of a battle the kingdom was lost
And all for the want of a horseshoe nail."

All of these authors speak of situations in which small events resulting big events and when such behavior is common, we call a situation chaotic. Chaotic situations are predictable in the short run but not in the long run. Until about 1975, few scientists realized that the simple situations they study could behave in this way. There were a few who were aware of such behavior, people such as Henri Poincaré, Steven Smale and his collaborators, Yasha Sinai and his colleagues in Moscow, Yoshisuke Ueda of Kyoto, Edward Lorenz of MIT, and several others. The great majority of scientists and engineers, however, felt that their systems had behavior that was highly predictable in the long run.

In the late 1970s there was a breakthrough in understanding. Many were aware that some very simple mathematical systems exhibited complicated behavior, but these were considered irrelevant anomalies. The breakthrough came about when scientists in different fields recognized that the chaotic behavior of these simple systems seemed to be reflected in the complex computer simulations of their research areas. Scientists (including mathematicians) suddenly began to realize that the systems they studied were chaotic for some choices of current or friction or flow rate or inflation rate or whatever was relevant to their investigation.

Our group is currently investigating a model created by the US National Weather Service. It models the weather for the whole planet. It is a computer program and to run it one must have estimates of the temperature, wind velocity, pressure and humidity at about 20,000 points on the Earth and at about 30 points above each of these (roughly 300 meters above the ground, 600 meters above the ground, 900 meters above the ground, etc.). If we provide an estimate of all these numbers, roughly 3,000,000 numbers all together, then the model will output an estimate of what the same numbers will be 10 minutes later. Send the new numbers back into the computer and one gets a prediction for the temperature, wind velocity, pressure, and humidity another 10 minutes later, or 20 minutes after starting. Repeatedly applying the program gives estimates as far in the future as one wishes. It is not perfect and depends strongly on having good estimates of the 3,000,000 numbers. If these estimates are only slightly incorrect, the errors will be magnified upon iteration of the model and soon the model will lose its predictive power.

Because it is so difficult to understand the properties of this complicated model, we often examine extremely simplified models such as

$x_{n+1} = 3.9\, x_n (1 - x_n)$. This model only depends on one number, x, instead of three million. Imagine that the number x (between 0 and 1) represents the state of a system now and $3.9\, x\, (1 - x)$ represents the state of the system after the elapse of one time period. If the simplified model had real meaning, we would not know x exactly. If we choose any tiny interval of values and apply the function to each number in this interval, we get a new "image" interval, and if we repeat the process we get a new image interval. As we repeat the process the sizes of the intervals usually increase, roughly doubling in length from application to application. Growing and growing and growing. Start with any interval of numbers between 0 and 1. Eventually you will get a big interval, that is, an interval whose length is at least 1/2.

At least that is what we observe, though there is no proof of the fact that it holds true for all tiny intervals and there almost certainly never will be. We do know that we can change the number 3.9 by arbitrarily small amounts and the result will be false. There will be some tiny intervals that do not grow ever larger. To prove the result for 3.9 we would have to find a relevant property of 3.9 that distinguishes it from the numbers arbitrarily close to it, numbers for which the expansion property is false.

This behavior is related to predictability. If there is an initial point that you apply the map repeatedly to, perhaps 1000 times, and you only know that the point lies in some tiny interval, then you must apply the map to all the points of the interval. Since the image of the tiny interval after 1000 applications will be huge and all you know is that the point is in that interval, you have little ability to predict.

We study this simple process, namely replacing x by $3.9\, x\, (1 - x)$ because it is a window into the world of chaotic phenomena. The difficulty of establishing results for this process hints at how difficult it is to establish results for more complicated processes.

Our study of the weather is aimed at using chaos indirectly to get a better estimate of what the weather is now. Less error now results in less error in a prediction for a few days from now. We are also using the ideas of chaos to find better ways to determine the sequence of letters (ACGTs) in the DNA in living species. Japan has for example reported a draft sequence for rice. Two groups in the United States reported obtaining a draft of the sequence for humans. Much work remains. We believe that the ideas of chaos that are similar to what we use for the weather will allow us to determine more accurate drafts of sequences at negligible cost.

Determining that a process is chaotic is usually not good news, but it is important knowledge for anyone who must coexist with the process.