Fractals and the measurement of roughness

My life work largely consists in fractal geometry, which I originated, developed in many directions, and named. This field overlaps the theory of chaos and can be viewed as the beginning of a theory of roughness. Being born in Eastern Europe gave me a burning ambition. A timely move to France saved my life and provided an idiosyncratic but outstanding culture in mathematics, science, and art. A later move the USA allowed this ambition and culture to flower.

At age twenty, my grades earned me the right to a smooth and comfortable future but on the condition of limiting myself to abstractions. Geometry and science have focused on smooth and simple phenomena. However, I rebelled against formalism, wanted to take advantage of the visual skills and imagination with which I was blessed, and was fascinated by the roughness and bottomless complication of raw nature. I chose therefore to forsake all beaten paths of science and was drawn to territory where no one had dwelt before. As a maverick scientist, I had a very lonely and rough ride. But it has been rewarding and, starting with my doctoral dissertation fifty years ago, it made me a pioneer of the field of complexity.

A few tools and concepts of fractal geometry had been developed earlier for diverse purposes quite different from mine. They arose within unabashed mathematical esoterica that were devised from 1875 to 1925, when mathematicians began to turn their backs to the world and celebrate theories unrelated to what we see and feel. I turned them around, increased their number and versatility, and made them concrete and useful in many ways. I also found gradually that, in other guises, fractality had been an unrecognized part of human experience since the dawn of humanity. Today, the world of fractals may be perceived as a ring from art back to art, through mathematics, finance, many sciences, and many corners of engineering.

Why is Euclid's standard geometry often described as grey, dull, cold, and dry? One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line. More generally, everyday patterns of Nature are far too irregular and fragmented to be accounted for by standard geometry. The level of complexity of much in Nature is not simply higher but altogether different. For practical purposes, natural patterns show an infinite number of distinct scales of length.

Identical remarks apply to many aspects of culture, that is, patterns revealed or created by Man. They range from the geometry of chance and deterministic chaos, to the shape of financial charts and to traffic on the internet - not to mention diverse aspects of painting, architecture, and music.

More specifically, Poincare remarked that there are questions that one chooses to ask and other questions that ask themselves. The latter can be mundane. The first few demonstrate that a question that has been without response tends to be abandoned to children.

- How to measure and compare the roughness of ordinary objects such as broken stone, metal, glass, or rusted iron?
- How long is the coast of Britain?
- What shape is the Earth, more precisely a mountain, a coastline, a river, or a dividing line between two rivers' watersheds? That is, can the term "geometry" deliver what it seems to promise?
- How to define the speed of the wind during a storm?
- What shape is a cloud, a flame, or a welding?

- What is the density of galaxies in the universe?

To this list, other questions were added recently:

- How to characterize the boundary between two basins of attraction in a chaotic dynamical system?
- How to measure the variation of the flow of messages on the Internet?
- How to measure the volatility of the prices quoted on financial markets?
- How to characterize the boundary of a plane random walk?

These questions challenge us to change our view of nature, to tame forms that standard geometry considers as formless. Responding broadly and specifically to this challenge, I conceived and developed a new geometry of nature and culture. Fractals satisfy a form of invariance called "scaling" which expresses that the degree of their irregularity and/or fragmentation is identical at all scales. Fractals can be curves, surfaces, or disconnected "dusts," and some are so oddly shaped that we formerly lacked terms for them. For example, Earth's relief and real physical fractures are fractal surfaces. Price records are multifractal functions. I showed how to imitate the mountains, the clouds and the price records. I also showed how to create wild and magical new shapes and provided mathematics with many difficult and inspiring questions.

Where does fractality stand among the sciences? Many sciences arose directly from the desire to describe and understand the basic messages, that the brain receives from the senses. In most instances, three stages can be distinguished reasonably clearly: elaboration of a rich descriptive vocabulary, elaboration of a "narrative" with a reassuring message of how everything holds together, and a proper science.

Thus, visual signals led to the vocabulary of bulk and shape, and of brightness and color; auditory signals, to the vocabulary of loudness and pitch. Optics and acoustics arose when the notions behind those words (for example, those known to musicians since time immemorial) were made quantitative. Similarly, the senses of heavy versus light and fast versus slow led to mechanics, and the sense of hot versus cold led to the theory of heat. Unquestioned proper measures of mass and size mark early writing and the dawn of history; temperature, which is a proper measure of uniform hotness, dates back to Galileo. Against this background, the sense of smooth versus rough, which a priori equally essential, was neglected.

The first step that transformed a mess of sensations into a quantitative science was about the same for many sciences. While the science of sound never claimed to be complete, it went far with healthy opportunism. Side-stepping the hard questions, it long identified pure sounds in song and music. The idealized sound of string instruments proved to be an "icon" that is at the same time reasonably realistic, mathematically manageable, and leading to a quantitative measurement. It clarifies even those facts it fails to characterize or explain. More generally, in the long road from raw sensation to science, a key moment was marked by the successful identification of a proper "compromise" between simplicity and breadth of applicability. Fractal geometry is the first agreed-upon quantity able to measure pure roughness the way temperature measures uniform hotness.

The Phenomena of Chaotic Dynamics

Chaos is an eternal human truth that we all know: small disturbances or minor events lead to large events. In Dostoevsky's *The Brothers Karamazov* (1879) an old priest relates an important truth:

"A touch in one place sets up movement at the other end of the Earth."

In 1972 Lorenz asked:

"Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

Benjamin Franklin described how a cascade of increasingly important events might emerge from a chance event:

"For the want of a nail the shoe was lost
For the want of a shoe the horse was lost.
For the want of a horse a rider was lost.
For the want of a rider the battle was lost.
For the want of a battle the kingdom was lost
And all for the want of a horseshoe nail."

All of these authors speak of situations in which small events resulting big events and when such behavior is common, we call a situation chaotic. Chaotic situations are predictable in the short run but not in the long run. Until about 1975, few scientists realized that the simple situations they study could behave in this way. There were a few who were aware of such behavior, people such as Henri Poincaré, Steven Smale and his collaborators, Yasha Sinai and his colleagues in Moscow, Yoshisuke Ueda of Kyoto, Edward Lorenz of MIT, and several others. The great majority of scientists and engineers, however, felt that their systems had behavior that was highly predictable in the long run.

In the late 1970s there was a breakthrough in understanding. Many were aware that some very simple mathematical systems exhibited complicated behavior, but these were considered irrelevant anomalies. The breakthrough came about when scientists in different fields recognized that the chaotic behavior of these simple systems seemed to be reflected in the complex computer simulations of their research areas. Scientists (including mathematicians) suddenly began to realize that the systems they studied were chaotic for some choices of current or friction or flow rate or inflation rate or whatever was relevant to their investigation.

Our group is currently investigating a model created by the US National Weather Service. It models the weather for the whole planet. It is a computer program and to run it one must have estimates of the temperature, wind velocity, pressure and humidity at about 20,000 points on the Earth and at about 30 points above each of these (roughly 300 meters above the ground, 600 meters above the ground, 900 meters above the ground, etc.). If we provide an estimate of all these numbers, roughly 3,000,000 numbers all together, then the model will output an estimate of what the same numbers will be 10 minutes later. Send the new numbers back into the computer and one gets a prediction for the temperature, wind velocity, pressure, and humidity another 10 minutes later, or 20 minutes after starting. Repeatedly applying the program gives estimates as far in the future as one wishes. It is not perfect and depends strongly on having good estimates of the 3,000,000 numbers. If these estimates are only slightly incorrect, the errors will be magnified upon iteration of the model and soon the model will lose its predictive power.

Because it is so difficult to understand the properties of this complicated model, we often examine extremely simplified models such as

$x_{n+1} = 3.9\, x_n(1 - x_n)$. This model only depends on one number, x, instead of three million. Imagine that the number x (between 0 and 1) represents the state of a system now and $3.9\, x\ (1 - x)$ represents the state of the system after the elapse of one time period. If the simplified model had real meaning, we would not know x exactly. If we choose any tiny interval of values and apply the function to each number in this interval, we get a new "image" interval, and if we repeat the process we get a new image interval. As we repeat the process the sizes of the intervals usually increase, roughly doubling in length from application to application. Growing and growing and growing. Start with any interval of numbers between 0 and 1. Eventually you will get a big interval, that is, an interval whose length is at least 1/2.

At least that is what we observe, though there is no proof of the fact that it holds true for all tiny intervals and there almost certainly never will be. We do know that we can change the number 3.9 by arbitrarily small amounts and the result will be false. There will be some tiny intervals that do not grow ever larger. To prove the result for 3.9 we would have to find a relevant property of 3.9 that distinguishes it from the numbers arbitrarily close to it, numbers for which the expansion property is false.

This behavior is related to predictability. If there is an initial point that you apply the map repeatedly to, perhaps 1000 times, and you only know that the point lies in some tiny interval, then you must apply the map to all the points of the interval. Since the image of the tiny interval after 1000 applications will be huge and all you know is that the point is in that interval, you have little ability to predict.

We study this simple process, namely replacing x by $3.9\, x\ (1 - x)$ because it is a window into the world of chaotic phenomena. The difficulty of establishing results for this process hints at how difficult it is to establish results for more complicated processes.

Our study of the weather is aimed at using chaos indirectly to get a better estimate of what the weather is now. Less error now results in less error in a prediction for a few days from now. We are also using the ideas of chaos to find better ways to determine the sequence of letters (ACGTs) in the DNA in living species. Japan has for example reported a draft sequence for rice. Two groups in the United States reported obtaining a draft of the sequence for humans. Much work remains. We believe that the ideas of chaos that are similar to what we use for the weather will allow us to determine more accurate drafts of sequences at negligible cost.

Determining that a process is chaotic is usually not good news, but it is important knowledge for anyone who must coexist with the process.